THE FREE ABELIAN TOPOLOGICAL GROUP AND THE FREE LOCALLY CONVEX SPACE ON THE UNIT INTERVAL †

Arkady G. Leiderman, [1] Sidney A. Morris
and Vladimir G. Pestov [2]

Ben-Gurion University of the Negev, P.O. Box 653, Beer-Sheva, Israel
University of Wollongong, Wollongong, N.S.W. 2522, Australia
Victoria University of Wellington, P.O. Box 600, Wellington, New Zealand

Abstract. We give a complete description of the topological spaces X such that the free abelian topological group $A(X)$ embeds into the free abelian topological group $A(I)$ of the closed unit interval. In particular, the free abelian topological group $A(X)$ of any finite-dimensional compact metrizable space X embeds into $A(I)$. The situation turns out to be somewhat different for free locally convex spaces. Some results for the spaces of continuous functions with the pointwise topology are also obtained. Proofs are based on the classical Kolmogorov's Superposition Theorem.

§1. Introduction

The following natural question arises as a part of the search for a topologized version of the Nielsen-Schreier subgroup theorem. Let X and Y be completely regular topological spaces; in which cases the free (free abelian) topological group over X can be embedded as a topological subgroup into the free (free abelian) topological group over Y? This problem has been treated for a long time [4, 10, 12-17, 21-23, 25, 28], ever since it became clear that in general a topological subgroup of a free (free abelian) topological group need not be topologically free [8, 4, 10]. Recently a complete answer was obtained in the case where X is a subspace of Y and the embedding of free topological groups extends the embedding of spaces [35]. However, we are interested in the existence of an embedding which is not necessarily a "canonical" one. Among the most notable achievements, there are certain sufficient conditions for a subgroup of a free topological group to be topologically free [4, 22] and the following results.

1991 *Mathematics Subject Classification.* 22A05, 55M10, 46A03.

Key words and phrases. Free abelian topological groups, free locally convex spaces, spaces of continuous functions, dimension, basic functions, Kolmogorov's Superposition Theorem.

† Dedicated to the memory of Eli Katz.

[1] Research Supported by the Israel Ministry of Science.

[2] Partially supported by the Internal Grants Committee of the Victoria University of Wellington.

Theorem 1.1 [14]**.** *If X is a closed topological subspace of the free topological group $F(I)$ then the free topological group $F(X)$ is a closed topological subgroup of $F(I)$, where I is the closed unit interval.* $\square$

Corollary 1.2 [22]**.** *If X is a finite-dimensional metrizable compact space then $F(X)$ is a closed topological subgroup of $F(I)$.* $\square$

The abelian case proved to be more difficult, and the following is the strongest result known to date.

Theorem 1.3 [12]**.** *If X is a countable CW-complex of dimension n, then the free abelian topological group on X is a closed subgroup of the free abelian topological group on the closed ball B^n.* $\square$

Corollary 1.4 [13]**.** *$A(\mathbb{R})$ embeds into $A(I)$ as a closed topological subgroup.* $\square$

It is known [29] that the covering dimension of any two free topological bases in a free (abelian) topological group is the same; this result is similar to the well-known property of free bases of a discrete free (abelian) group having the same cardinality, called the rank of the group. Since the rank of a subgroup of a free abelian group cannot exceed the rank of the group itself, it was conjectured [15, 20] that the dimension of a topological basis of a topologically free subgroup of a free abelian topological group $A(X)$ cannot exceed *dim* X. It remained even unclear whether the group $A(I^2)$ embeds into $A(I)$ [15].

In this paper we prove that if X is a completely regular space then the free abelian topological group $A(X)$ embeds into $A(I)$ as a topological subgroup if and only if X is a submetrizable k_ω-space such that every compact subspace of X is finite-dimensional. Another characterization: X is homeomorphic to a closed topological subspace of the group $A(I)$ itself. In particular, if X is a compact metrizable space of finite dimension, then $A(X)$ embeds into $A(I)$. Thus, the analogy with the non-abelian case is complete. We also study the problem of embedding the free locally convex space $L(X)$ into the free locally convex space $L(I)$ and characterize those k_ω-spaces X admitting such an embedding. Paradoxically, such spaces X are just all compact metrizable finite-dimensional spaces. In particular, the free LCS $L(\mathbb{R})$ does not embed into $L(I)$. Our results provide answers to a number of open problems from [20, 15, 27].

§2. Preliminaries

Definition 2.1 [19, 8, 20]**.** Let X be a topological space. The *(Markov) free abelian topological group over* X is a pair consisting of an abelian topological group $A(X)$ and a topological embedding $X \hookrightarrow A(X)$ such that every continuous mapping f from X to an abelian topological group G extends uniquely to a continuous homomorphism $\bar{f} : A(X) \to G$. $\square$

If X is a completely regular topological space then the free abelian topological group $A(X)$ exists and is algebraically free over the set X [19, 8, 20]. A topological space X is called a k_ω-space [18, 13-17] if there exists a so-called k_ω-decomposition $X = \cup_{n\in\mathbb{N}} X_n$, where all X_n are compact, $X_n \subset X_{n+1}$ for $n \in \mathbb{N}$, and a subset $A \subset X$ is closed if and only if all intersections $A \cap X_n$, $n \in \mathbb{N}$, are closed. All locally convex spaces (LCS) in this paper are real.

Definition 2.2 [19, 1, 31, 6, 7, 34]**.** Let X be a topological space. The *free locally convex space over* X is a pair consisting of a locally convex space $L(X)$ and a topological embedding $X \hookrightarrow L(X)$ such that every continuous mapping f from X to a locally convex space E extends uniquely to a continuous linear operator $\bar{f} : L(X) \to E$. $\square$

If X is a completely regular topological space then the free locally convex space $L(X)$ exists; the set X forms a Hamel basis for $L(X)$ [31, 6, 7, 34]. The identity mapping $id_X : X \to X$ extends to a canonical continuous homomorphism $i : A(X) \to L(X)$.

Theorem 2.3 [33]**.** *The canonical homomorphism $i : A(X) \hookrightarrow L(X)$ is an embedding of $A(X)$ into the additive topological group of the LCS $L(X)$ as a closed additive topological subgroup.* $\square$

In what follows, we will often identify $A(X)$ with a subgroup of $L(X)$ in the above canonical way. Denote by $L_p(X)$ the free locally convex space $L(X)$ endowed with the weak topology.

Theorem 2.4 [6, 7]**.** *Let X be a completely regular space. The canonical mapping $X \hookrightarrow L_p(X)$ is a topological embedding, and every continuous mapping f from X to a locally convex space E with the weak topology extends uniquely to a continuous linear operator $\bar{f} : L_p(X) \to E$.* $\square$

The weak dual space to $L(X)$ is canonically isomorphic to the space $C_p(X)$ of all continuous real-valued functions on X with the topology of pointwise (simple) convergence. The spaces $L_p(X)$ and $C_p(X)$ are in duality. Denote by $C_k(X)$ the space of continuous functions endowed with the compact-open topology. A topological space X is called *Dieudonné complete* [5] if its topology is induced by a complete uniformity. For example, every Lindelöf space is Dieudonné complete. In particular every k_ω-space is Dieudonné complete.

Theorem 2.5 (Arhangel'skiĭ [3])**.** *Let X and Y be Dieudonné complete spaces. If a linear mapping $C_p(X) \to C_p(Y)$ is continuous then it is continuous as a mapping $C_k(X) \to C_k(Y)$.* $\square$

The space $L(X)$ admits a canonical continuous monomorphism

$$L(X) \hookrightarrow C_k(C_k(X))$$

Theorem 2.6 (Flood [6, 7], Uspenskiĭ [34])**.** *If X is a k-space then the monomorphism $L(X) \hookrightarrow C_k(C_k(X))$ is an embedding of locally convex spaces.* $\square$

Let X be a topological space. A collection of continuous functions $h_1, \dots, h_m$ on X assuming their values in the closed unit unterval $I = [0, 1]$ is called *basic* [26, 32] if every real-valued continuous function f on X can be represented as a sum $\sum_{i=1}^{n} g_i \circ h_i$ of compositions of basic functions with some continuous functions $g_i \in C(I)$.

2.7. Kolmogorov's Superposition Theorem [11]**.** *The finite-dimensional cube I^n has a finite basic family of continuous real-valued functions.* $\square$

Let us recall that for compact metrizable spaces all three main concepts of di-

[5]. The following result is of crucial importance for us; it is an immediate corollary of the Kolmogorov's Superposition Theorem, the Menger-Nöbeling Theorem on embeddability of separable metric spaces of dimension $\leq n$ into $\mathbb{R}^{2n+1}$, and the Tietze-Urysohn Extension Theorem [5].

Corollary 2.8 (Ostrand [26]). *Let X be a finite-dimensional compact metrizable space. Then there exists a finite basic family of continuous functions on X.* $\square$

For an exact upper bound on the cardinality of a basic family of continuous functions on a space X of dimension n, see [32]; however, we do not need it.

§3. Auxiliary constructions

Lemma 3.1. *Consider a commutative diagram of Banach spaces and surjective continuous linear mappings:*

$$\begin{tikzcd}
E_1 \arrow[d, "\pi_1"'] & E_2 \arrow[l, "r_1"'] \arrow[d, "\pi_2"'] & E_3 \arrow[l, "r_2"'] \arrow[d, "\pi_3"'] & \dots \arrow[l, "r_3"'] & E_n \arrow[l, "r_{n-1}"'] \arrow[d, "\pi_n"'] & \dots \arrow[l, "r_n"'] \\
F_1 & F_2 \arrow[l, "q_1"'] & F_3 \arrow[l, "q_2"'] & \dots \arrow[l, "q_3"'] & F_n \arrow[l, "q_{n-1}"'] & \dots \arrow[l, "q_n"']
\end{tikzcd}$$

Denote by $E = \varprojlim E_n$ and $F = \varprojlim F_n$ the Fréchet spaces projective limits of corresponding inverse sequences, and by $\pi : E \to F$ the projective limit of the mappings π_n, $n \in \mathbb{N}$. Then every compact subspace $K \subset F$ is an image under the mapping π of a compact subspace of E.

Proof. Let K be a compact subspace of F. Let $K_n = q_n(K)$ for all $n \in \mathbb{N}$. According to the Michael Selection Theorem (Th. 1.4.9 in [36]), there exists a compact subspace $C_1 \subset E_1$ such that $\pi_1(C_1) = K_1$. Assume now that for all $k \leq n$ we have chosen compact subspaces $C_k \subset E_k$ such that $\pi_k(C_k) = K_k$ and $r_{k-1}(C_k) = C_{k-1}$. Consider the mapping $< r_n, \pi_{n+1} >: x \mapsto (r_n(x), \pi_{n+1}(x))$ from E_{n+1} to $E_n \times F_{n+1}$. The subset $Q_n = \{(y, z) : y \in C_n, z \in K_{n+1}, q_n(z) = \pi_n(y)\}$ of the space $E_n \times F_{n+1}$ is compact, and is contained in the Banach space image of the continuous linear mapping $< r_n, \pi_{n+1} >$. Therefore, by the Michael Selection Theorem, there exists a compact subset $C_{n+1} \subset E_{n+1}$ such that $< r_n, \pi_{n+1} > (C_{n+1}) = Q_n$. Consequently, $r_n(C_{n+1}) = C_n$, and $q_{n+1}(C_{n+1}) = K_{n+1}$, which completes the recursion step. Finally, put $C = \varprojlim C_n$; this subset of E is compact, and the property $K \subset \varprojlim K_n$ implies that $\pi(C) = K$. $\square$

Lemma 3.2. *Let X and Y be k_ω-spaces. Let $h : L_p(X) \to L_p(Y)$ be an embedding of locally convex spaces. Then h is an embedding of locally convex space $L(X)$ into $L(Y)$ as well.*

Proof. As a corollary of the Hahn-Banach theorem, the dual linear map $h^* : C_p(Y) \to C_p(X)$ to the embedding h is a continuous surjective homomorphism. Theorem 2.5 says that h^* remains continuous with respect to the compact-open topologies on both spaces, and by virtue of the Open Mapping Theorem, $h^* : C_k(Y) \to C_k(X)$ is open. Since for every compact subset $C \subset X$ the elements of the image $h(C)$ are contained in the linear span of a compact subset of Y [3], one can choose k_ω-decompositions $X = \cup_{n=1}^{\infty} X_n$ and $Y = \cup_{n=1}^{\infty} Y_n$ in such a way that for every $n \in \mathbb{N}$ one has $h(sp\ X_n) \subset sp\ Y_n$. It is easy to see that the restrictions

surjections, and that $C(Y) = \varprojlim C_k(Y_n)$ and $C(X) = \varprojlim C_k(X_n)$. Denote for each $n \in \mathbb{N}$ by π_n the restriction $h^*|_{C_k(Y_n)}$. The conditions of Lemma 3.1 are fulfilled, and therefore every compact subset $K \subset C_k(X)$ is an image under the mapping h^* of a suitable compact subset of $C_k(Y)$. Therefore, the continuous linear map h^{**} dual to h^* from the space $C_k(C_k(X))$ to $C_k(C_k(Y))$ is an embedding of $C_k(C_k(X))$ into $C_k(C_k(Y))$ as a locally convex subspace. Since the restriction of h^{**} to $L(X)$ is h, the desired statement follows from Theorem 2.6. $\square$

Lemma 3.3. *Let X be a compact space and let Y be a closed subspace of X. Denote by π the quotient mapping from X to X/Y. Let $f_k,\ k = 1, \ldots, n$ be continuous functions on X such that their restrictions to Y form a basic family for Y, and let $g_i,\ i = 1, \ldots, m$ be a basic family of functions on X/Y. Then the family of functions $f_1, \ldots, f_n, g_1 \circ \pi, \ldots, g_m \circ \pi$ is basic for X.*

Proof. Let $f : X \to \mathbb{R}$ be a continuous function. For a family of continuous functions $h_1, \ldots, h_n \in C(I)$, the restriction $f|_Y$ is represented as $\sum_{k=1}^n h_k \circ (f_k|_Y) = (\sum_{k=1}^n h_k \circ f_k)|_Y$. Denote by $g : X \to \mathbb{R}$ the continuous function $f - \sum_{k=1}^n h_k \circ f_k$; since the restriction $g|_Y \equiv 0$, the function g factors through the mapping π; that is, there exists a continuous function $h : X/Y \to I$ with $g = h \circ \pi$. For some collection $s_1, \ldots, s_m$ of continuous functions on I one has $h = \sum_{i=1}^m s_i \circ g_i$, which means that $g = \sum_{i=1}^m s_i \circ g_i \circ \pi$. Finally, one has

$$f = \sum_{k=1}^{n} h_k \circ f_k + \sum_{i=1}^{m} s_i \circ g_i \circ \pi,$$

as desired. $\square$

A topological space X is called *submetrizable* if it admits a continuous one-to-one mapping into a metrizable space.

Lemma 3.4. *Let X be a submetrizable k_ω-space with k_ω-decomposition $X = \cup_{n \in \mathbb{N}} X_n$ such that every subspace X_n is finite-dimensional. Then there exists an embedding of locally convex spaces $\bar{F} : L_p(X) \hookrightarrow L_p(Y)$, where Y is the disjoint sum of countably many copies of the closed unit interval I, such that $\bar{F}(A(X)) \subset A(Y)$.*

Proof. Let $X = \cup_{n \in \mathbb{N}} X_n$ be a k_ω decomposition of X with $X_n \subset X_{n+1}$, for all $n \in \mathbb{N}$. Since every $X_n,\ n \in \mathbb{N}$ is a finite-dimensional metrizable compact space, then for any $n \in \mathbb{N}$ so is the quotient space X_{n+1}/X_n, and one can choose inductively, using Ostrand's Corollary 2.8 and Lemma 3.3, a countable family of continuous functions $f_{n,i},\ n \in \mathbb{N},\ i = 1, \ldots, k_n,\ k_n \in \mathbb{N}$ from X to I such that for each $n \in \mathbb{N}$ the following are true:

1. the collection $f_{m,i},\ i = 1, \ldots, k_m,\ m = 1, \ldots, n$, is basic for X_n;
2. $f_{n+1,i}|_{X_n} \equiv 0$ for all $i = 1, \ldots, k_{n+1}$. Denote the above family of functions $f_{n,i}$ by $\mathcal{F}$, and let $Y = \oplus_{f \in \mathcal{F}} I_f$ be the disjoint sum of countably many copies of the closed unit interval I. For every $f \in \mathcal{F}$ denote by 0_f the left endpoint of the closed interval I_f regarded as an element of the free abelian group $A(Y)$. Define a mapping, F, from X to the free abelian group $A(Y)$ by letting

$$F(x) = \sum_{i=1,\ldots,k_1} f_{1,i}(x) + \sum_{n \geq 2,\ i=1,\ldots,k_n} (f_{n,i}(x) - 0_{n,i})$$

for each $x \in X$. The mapping F is properly defined, because the first sum is finite and in the second sum all but finitely many terms are vanishing in the

free abelian group $A(Y)$, for every $x \in X$. The restriction of F to every X_n is continuous if being considered as a mapping to the free abelian topological group $A(Y)$, which fact follows from continuity of each mapping $f_{m,i} : X_n \to I_{f_{m,i}} \subset Y,\ m \leq n,\ i = 1, \dots, k_m$ and the continuity of subtraction and addition in $A(Y)$. Therefore the mapping $F : X \to A(Y)$ is continuous. If being viewed as a continuous mapping from X to the locally convex space $L_p(Y)$, it extends to a continuous linear operator $\bar{F} : L_p(X) \to L_p(Y)$. Let $h : X \to \mathbb{R}$ be a continuous function. We will show that there exists a continuous linear functional $\bar{h}$ on the linear subspace $\bar{F}(L_p(X))$ such that $\bar{h} \circ F|_X = h$. It would mean that $\bar{F}(L_p(X))$ is isomorphic to $L_p(X)$, as desired. Construct recursively, making use of the properties 1 and 2 above, a countable family of continuous functions $h_{n,i},\ i = 1 \dots, k_n,\ n \in \mathbb{N}$ from I to $\mathbb{R}$ such that for every $n \in \mathbb{N}$ and for all $x \in X_n$,

$$h(x) = \sum_{i=1,\dots,k_m,\ m \leq n} (h_{m,i} \circ f_{m,i})(x)$$

Let us recall that $f_{n,i}|_{X_1} \equiv 0$ for all $n \geq 2$ and $i = 1, \dots, k_n$. It is easy to deduce inductively from this fact that for any $n \geq 2$

$$\sum_{i=1,\dots,k_n} h_{n,i}(0) = 0$$

Define a continuous mapping H from Y to I by letting $H(y) = h_{n,i}(y)$, if $y \in I_{n,i},\ n \in \mathbb{N}$. Extend H to a continuous linear functional $\bar{H} : L_p(Y) \to \mathbb{R}$ and denote its restriction to $\bar{F}(L_p(X))$ by $\bar{h}$. We claim that $\bar{h} \circ F|_X = h$, or, which is the same, that for every $n \in \mathbb{N}$ one has $\bar{h} \circ F|_{X_n} = h$. Indeed, for an arbitrary $x \in X_n$ one has:

$$(\bar{h} \circ F)(x) = \bar{H}(F(x)) = \bar{H}\Big(\sum_{i=1,\dots,k_1} f_{1,i}(x) + \sum_{2 \leq m \leq n,\ i=1,\dots,k_n} (f_{m,i}(x) - 0_{m,i})\Big)$$

$$= \sum_{i=1,\dots,k_1} H(f_{1,i}(x)) + \sum_{2 \leq m \leq n,\ i=1,\dots,k_n} H(f_{m,i}(x) - 0_{m,i})$$

$$= \sum_{i=1,\dots,k_m,\ m \leq n} (h_{m,i} \circ f_{m,i})(x) - \sum_{2 \leq m \leq n,\ i=1,\dots,k_n} h_{m,i}(0) = h(x) - 0 = h(x).$$

□

§4. Main results

Theorem 4.1. *For a completely regular space X the following are equivalent.*

(i) The free abelian topological group $A(X)$ embeds into $A(I)$ as a topological subgroup.

(ii) The free topological group $F(X)$ embeds into $F(I)$ as a topological subgroup.

(iii) X is homeomorphic to a closed topological subspace of $A(I)$.

(iv) X is homeomorphic to a closed topological subspace of $F(I)$.

(v) X is homeomorphic to a closed topological subspace of $\mathbb{R}^\infty$.

(vi) X is a k_ω-space such that every compact subspace of X is metrizable and finite-dimensional.

(vii) *X is a submetrizable k_ω-space such that every compact subspace of X is finite-dimensional.*

Proof. *(i)* $\Rightarrow$ *(iii)*: since the space X is Lindelöf (as a subspace of $A(I)$, see [2]) and hence Dieudonné complete, the group $A(X)$ is complete in its two-sided uniformity [33] and therefore closed in $A(I)$; but X is closed in $A(X)$. *(ii)* $\Leftrightarrow$ *(iv)*: see [14]. *(iii)* $\Leftrightarrow$ *(v)* $\Leftrightarrow$ *(iv)*: follows from the result of Zarichnyĭ [37]: the free topological group $F(I)$ and the free abelian topological group $A(I)$ are homeomorphic to open subsets of $\mathbb{R}^\infty$. *(v)* $\Rightarrow$ *(vi)*: the space $\mathbb{R}^\infty = \varinjlim \mathbb{R}^n$ is a k_ω-space such that every compact subspace of it is metrizable and finite-dimensional, and this property is inherited by closed subsets. *(vi)* $\Leftrightarrow$ *(vii)*: see [16]. *(vii)* $\Rightarrow$ *(i)*: Let X be a submetrizable k_ω space such that every compact subspace of X is finite-dimensional. According to Lemma 3.4, there exists an embedding of locally convex spaces $\bar{F}: L_p(X) \hookrightarrow L_p(Y)$, where Y is the disjoint sum of countably many copies of the closed unit interval I, such that $\bar{F}(A(X)) \subset A(Y)$. By virtue of Lemma 3.2, $\bar{F}$ is also an embedding of locally convex spaces $L(X) \hookrightarrow L(Y)$. Its restriction to $A(X)$ is an embedding of topological groups (Theorem 2.3). Now apply Corollary 1.4. □

Theorem 4.2. *For a k_ω-space X the following conditions are equivalent.*

(i) *The free locally convex space $L(X)$ embeds into $L(I)$ as a locally convex subspace.*

(ii) *The free locally convex space with the weak topology, $L_p(X)$, embeds into $L_p(I)$ as a locally convex subspace.*

(iii) *The space $C_p(X)$ is a quotient linear topological space of $C_p(I)$.*

(iv) *X is a finite-dimensional metrizable compact space.*

Proof. *(ii)* $\Leftrightarrow$ *(iii)*: just dual forms of the same statement about two locally convex spaces having their weak topology. *(i)* $\Rightarrow$ *(iv)*: Suppose X is a noncompact k_ω-space. Since (ii) and by the same token (iii) hold, then by virtue of Theorem 2.5 the Fréchet non-normable space $C_k(X)$ is an image of the Banach space $C_k(I)$ under a surjective continuous linear mapping, which is open by virtue of the Open Mapping Theorem – a contradiction. Now the space X, being compact, is contained in the subspace $sp_n(Y)$ of $L(Y)$ formed by all words of the reduced length $\leq n$ over Y for some $n \in \mathbb{N}$ [34]. But the space $sp_n(Y)$ is a union of countably many closed subspaces each of which is homeomorphic to a subspace of the n-th Tychonoff power of the space $\mathbb{R} \times [X \oplus (-X) \oplus \{0\}]$ [2]. Therefore, $sp_n(Y)$ is finite-dimensional. Finally, submetrizability of X follows from the same property of $L(I)$ (the latter space admits a continuous one-to-one isomorphism into the free Banach space over I, [1, 6, 7]). *(iv)* $\Rightarrow$ *(ii)*: it follows from Lemmas 3.4 and 3.3 that $L_p(X)$ embeds as a locally convex subspace into the free locally convex space in the weak topology over a disjoint sum of finitely many homeomorphic copies of the closed interval. The latter LCS naturally embeds into $L_p(I)$. *(ii)* $\Rightarrow$ *(i)*: apply Lemma 3.3. □

Remark 4.3. Surprising as it may seem, the free locally convex space $L(\mathbb{R})$ does not embed into $L(I)$, in spite of the existence of canonical embeddings $A(\mathbb{R}) \hookrightarrow L(\mathbb{R})$ and $A(I) \hookrightarrow L(I)$ and a (non-canonical one) $A(\mathbb{R}) \hookrightarrow A(I)$. It is another illustration to the well-known fact that not every continuous homomorphism to the additive group of reals from a closed additive subgroup of an (even normable) LCS extends to a continuous linear functional on the whole space. Such a misbehaviour is also to blame — at least partly — for apparent lack of progress in attempts to make the Pontryagin-van Kampen duality work for free abelian topological groups [24, 30].

Remark 4.4. The problem of characterization of covering dimension of a completely regular space X in terms of the linear topological structure of the space $C_p(X)$ still remains open (cf. [9]). However, now we can describe those metrizable compact spaces having finite dimension.

Corollary 4.5. *A metrizable compact space X is finite-dimensional if and only if the space $C_p(X)$ is a quotient linear topological space of $C_p(I)$.* $\square$

Perhaps, the dimension of X can be described in terms referring to the linear topological structure of the space $C_p(X)$ with the help of a characterization of dimension in the language of basic functions due to Sternfeld [32].

Remark 4.6. Our results also provide answers to three problems from the book *Open Problems in Topology* [27].

PROBLEM 511. *Is $A(I^2)$ topologically isomorphic with a subgroup of $A(I)$?*

Yes (cf. Theorem 4.1).

PROBLEM 1046. *Assume that $C_p(X)$ can be mapped by a linear continuous mapping onto $C_p(Y)$. Is it true that $\dim Y \leq \dim X$? What if X and Y are compact?*

PROBLEM 1047. *Assume that $C_p(X)$ can be mapped by an open linear continuous mapping onto $C_p(Y)$. Is it true that $\dim Y \leq \dim X$? What if X and Y are compact?*

No, in all four cases (cf. Corollary 4.5).

Acknowledgment

The third author (V.G.P.) thanks the University of Wollongong for hospitality extended in August-September 1992.

References

[1] R. Arens and J. Eells, *On embedding uniform and topological spaces*, Pacific J. Math. **6** (1956), 397–403.

[2] A.V. Arhangel'skiĭ, *Relations among invariants of topological groups and their subspaces*, Russ. Math. Surveys **35** (1980), 1–23.

[3] A.V. Arhangel'skiĭ, *On linear homeomorphisms of function spaces*, Soviet Math. Dokl. **25** (1982), 852–855.

[4] R. Brown and J.P.L. Hardy, *Subgroups of free topological groups and free topological products of topological groups*, J. London Math. Soc. (2) **10** (1975), 431–440.

[5] R. Engelking, *General Topology*, PWN, Warczawa, 1977.

[6] J. Flood, *Free topological vector spaces*, Ph.D. thesis, Australian National University, Canberra, 1975, 109 pp.

[7] J. Flood, *Free locally convex spaces*, Dissert. Math. **CCXXI** (1984), PWN, Warczawa.

[8] M.I. Graev, *Free topological groups*, Amer. Math. Soc. Transl. **35** (1951), 61 pp.

[9] S.P. Gul'ko, *On uniform homeomorphisms of spaces of continuous functions*, Proc. Baku Intern. Topol. Conf. (1987), Trudy Mat. Inst. Steklova **193** (1990), (in Russian).

[10] D.C. Hunt and S.A. Morris, *Free subgroups of free topological groups*, Proc. Second Internat. Conf. Theory of Groups (Canberra 1973), Lect. Notes Math. **372**, Springer-Verlag, Berlin, 1974, pp. 377–387.

[11] A.N. Kolmogorov, *On the representation of continuous functions of many variables*, Amer. Math. Soc. Transl. (2) **28** (1963), 55–61.

[12] E. Katz and S.A. Morris, *Free abelian topological groups on countable CW-complexes*, Bull. Austral. Math. Soc. **41** (1990), 451–456.

[13] E. Katz, S.A. Morris and P. Nickolas, *A free subgroup of the free abelian topological group on the unit interval*, Bull. London Math. Soc. **14** (1982), 399–402.

[14] E. Katz, S.A. Morris and P. Nickolas, *Characterization of bases of subgroups of free topological groups*, J. London Math. Soc. (2) **27** (1983), 421–426.

[15] E. Katz, S.A. Morris and P. Nickolas, *Free abelian topological groups on spheres*, Quart. J. Math. Oxford (2) **35** (1984), 173–181.

[16] E. Katz, S.A. Morris and P. Nickolas, *Free subgroups of free abelian topological groups*, Math. Proc. Camb. Phil. Soc. **100** (1986), 347–353.

[17] E. Katz, S.A. Morris and P. Nickolas, *Free abelian topological groups and adjunction spaces*, J. Pure and Appl. Algebra **68** (1990), 209–214.

[18] J. Mack, S.A. Morris and E.T. Ordman, *Free topological groups and the projective dimension of a locally compact abelian group*, Proc. Amer. Math. Soc. **40** (1973), 303–308.

[19] A.A. Markov, *On free topological groups*, Doklady Akad. Nauk SSSR **31** (1941), 299–301 (in Russian).

[20] S.A. Morris, *Free abelian topological groups*, Categorical Topology, Proc. Conference Toledo, Ohio, 1983, Heldermann-Verlag, 1984, pp. 375–391.

[21] S.A. Morris and V.G. Pestov, *Open subgroups of free abelian topological groups*, Math. Proc. Camb. Phil. Soc. (submitted for publication), Res. rep. RP-92-98, Dept. Math., Victoria Univ. of Wellington, Sept. 1992, 5 pp.

[22] P. Nickolas, *A Schreier theorem for free topological groups*, Bull. Austral. Math. Soc. **13** (1975), 121–127.

[23] P. Nickolas, *Subgroups of the free topological group on* $[0,1]$, J. London Math. Soc. (2) **12** (1976), 199–205.

[24] P. Nickolas, *Reflexivity of topological groups*, Proc. Amer. Math. Soc. **65** (1977), 137–141.

[25] E. Nummela, *Uniform free topological groups and Samuel compactifications*, Topology and its Appl. **13** (1982), 77–83.

[26] P.A. Ostrand, *Dimension of metric spaces and Hilbert's problem 13*, Bull. Amer. Math. Soc. **71** (1965), 619–622.

[27] *Open Problems in Topology* (J. van Mill and G.M. Reed, eds.) (1990), North-Holland, Amsterdam.

[28] V.G. Pestov, *Some properties of free topological groups*, Moscow Univ. Math. Bull. **37** (1982), 46–49.

[29] V.G. Pestov, *The coincidence of the dimensions dim of l-equivalent topological spaces*, Soviet Math. Dokl. **26** (1982), 380–382.

[30] V.G. Pestov, *Free topological abelian groups and the Pontryagin duality*, Mosc. Univ. Math. Bull. **41** (1986), 1–4.

[31] D.A. Raĭkov, *Free locally convex spaces for uniform spaces*, Mat. Sb. (N.S.) **63** (1964), 582–590 (in Russian).

[32] Y. Sternfeld, *Dimension, superposition of functions and separation of points in compact metric spaces*, Israel J. Math. **50** (1985), 13–53.

[33] M.G. Tkachenko, *On completeness of free abelian topological groups*, Soviet Math. Dokl. **27** (1983), 341–345.

[34] V.V. Uspenskiĭ, *On the topology of free locally convex space*, Sov. Math. Dokl. **27** (1983), 781–785.

[35] V.V. Uspenskiĭ, *Free topological groups of metrizable spaces*, Math. USSR - Izvestiya **37** (1991), 657–680.

[36] J. van Mill, *Infinite-Dimensional Topology*, North-Holland Publ. Co., Amsterdam, 1989.

[37] M.M. Zarichnyĭ, *Free topological groups of absolute neighbourhood retracts and infinite-dimensional manifolds*, Soviet Math. Dokl. **26** (1982), 367–371.